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Coasian Dynamics in Repeated English Auctions*

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1 Introduction

Ronald Coase (1972) considered a monopolist in possession of a quantity of some durable good to which she attaches no value. Demand is such that all of the good could be sold at a positive price. Coase argues that (i) all of the good will therefore be sold eventually; (ii) since every potential buyer knows this, none will be prepared to pay more than any other buyer; and hence (iii) every unit will sell more-or-less instantly for a price equal to the valuation of the marginal buyer. This is the famous *Coase conjecture*. It reveals how the monopolist "competes with herself" – that is, later instantiations of the monopolist compete with earlier ones – so as to drive market profits down to their competitive level.

Coase's argument is easily adapted to the case of a seller with a single, indivisible unit of some durable good, facing a single potential buyer whose valuation is unknown to the seller (though it is commonly known to exceed the seller's). To do so, we reinterpret the demand curve faced by Coase's monopolist as the (inverse) decumulative distribution function describing the buyer's valuation. If the seller makes take-it-or-leave-it offers, Coase's conjecture implies that she will quickly come to accept that the best she can do is offer a price equal to the lowest valuation in the support of the valuation distribution, which the buyer will duly accept.

Of course, Coase's argument is informal, and step (ii) in particular is rather casual about the time costs of waiting for a lower price and the possibility that different buyers will have different incentives to wait. However, several subsequent papers have placed the argument on a sounder footing.[1] These papers show that, under certain conditions,[2] the following version of the Coase conjecture may be proved: as the time between seller offers vanishes, the seller's initial offer converges to the minimum value in the support of the valuation distribution.[3]

More recently still, McAfee and Vincent (1997) showed that the Coase conjecture may be extended to the case of n buyers whose unknown valuations are independent draws from some common distribution. They consider a scenario in which the seller holds a sequence of *sealed-bid auctions* with

[1] See the survey by Ausubel, Crampton and Deneckere (2002).

[2] One being that there exist an $\varepsilon > 0$ such that it is common knowledge that the buyer's valuation exceeds the seller's by at least ε – the so-called "gap" assumption.

[3] See Tirole (1988) for a nice introduction to the Coase conjecture and a sketch proof for the "no gap" case.

posted *reserve prices*, and show that a Coasian dynamic characterises the reserve price path.[4] They also demonstrate that first-price and second-price auctions are revenue-equivalent in this repeated auction game.

The purpose of the present paper is twofold. First, we show that the repeated *English* auction leads to a lower equilibrium path of reserve prices than the sealed-bid format.[5] Second, we show that, nevertheless, the English and sealed-bid formats are revenue equivalent.[6] It follows then from the analysis in McAfee and Vicnent (1997) that the usual Coasian logic applies to repeated English auctions.

The proof of the first result closely follows that in McAfee and Vincent (1997), which in turn is based on the classic argument in Fudenberg, Levine and Tirole (1985). However, the process of adaptation is useful for exposing the intuition behind the result. Sealed-bid auctions induce earlier bidding – *ceteris paribus* – because they attract fewer participants. With English auctions, once bidding starts, all bidders participate, dropping out only once their value is reached. With sealed-bid auctions, bidders cannot observe whether any other bidder has submitted a bid in the current round. By bidding early, a bidder in a sealed-bid auction can eliminate competition from rivals with lower valuations (that are nevertheless above the current reserve) who plan to wait and bid in subsequent rounds.

It is therefore not obvious that the two auction formats will yield the same revenue to the seller. On the one hand, bidders in sealed-bid auctions, facing the same path of reserve prices, will bid earlier than in English auctions, which is good for the seller. But on the other hand, the participation effect will reduce the expected price, which is bad. We show that, in equilibrium, the seller arranges matters so that these two effects exactly cancel each other. Moreover, the good will be allocated in exactly the same way. That is to say, if the maximum bidder value is v, the good will sell in the same period for the same expected price under either format. This provides a Revenue

[4] They too make the "gap" assumption.

[5] See Grant, Kajii, Menezes and Ryan (2006) for an analysis of repeated English auctions when bidders arrive randomly (eg, according to a Poisson process), and participate in only one auction round.

[6] Beckert (2006) compares sequential sealed-bid auctions with sequential take-it-or-leave-it pricing, for a two-period model. As shown in the 2002 Working Paper version, sequential sealed-bid auctions are revenue dominated by sequential take-it-or-leave-it pricing when players are sufficiently patient. We are unaware of any work that compares the revenue from the two mechanisms in an infinite period setting.

Equivalence result.

2 The repeated English auction

For ease of comparison, we adhere closely to the set-up in McAfee and Vincent (1997; henceforth MV97). A monopoly seller has a single indivisible good to which she attaches consumption value zero. There are n potential buyers. Each buyer's valuation is an independent realisation of a random variable with cumulative distribution function F. This distribution satisfies $F(1) = 0$, $F(v_H) = 1$ and has strictly positive and bounded density f on $[1, v_H]$. Moreover, we assume f is continuous at 1. Buyers and seller are risk neutral with identical rate of time preference ρ.

The seller holds a potentially infinite sequence of English auctions until a sale is achieved. Within each round she posts a reserve price. The rules of the auction oblige her to sell to the highest bidder if a *serious bid* (i.e., one weakly above the reserve) is received. Moreover, she has access to a limited commitment technology. She can commit not to sell in the current round if bidding fails to reach the reserve, but she cannot commit to cancel future auctions if the object remains unsold. The payoff to a bidder who fails to secure the object is assumed to be zero, no matter when the sale occurs.

There is no delay between auction rounds and the nominal duration of each auction is fixed at one unit of time. However, we assume that auctions have a "soft ending", so bidders always have an opportunity to respond to a new bid: there is no "last moment" in which a bid could be submitted so as to exclude counter-bids. Lemma 0(i) shows that optimal bidding strategies may be assumed to be independent of the elapsed time during auctions, so no-one expects this "soft ending" facility to be called upon.[7] Thus, if the object is expected to sell after t rounds for a price p to a bidder with valuation v, the buyer anticipates payoff $\delta^t (v - p)$ and the seller anticipates $\delta^t p$, where $\delta = e^{-\rho}$.

We denote this repeated auction game $\mathcal{G}$ and shall use the notion of *perfect Bayesian equilibrium (pBe)* to "solve" $\mathcal{G}$.

It is clear that *once a serious bid has been submitted*, everyone knows that they are in a *one-shot* auction, so each bidder will bid up to his valuation

[7] It may, of course, be called upon out of equilibrium. However, if off-equilibrium *sniping* of this sort were to take place, the game would end in the round in which it occurs. In such contingencies, all bidders bid their values, so these contingencies are covered by Lemma 0.

during the current round. It remains only to specify the contingencies in which a bidder *initiates* serious bidding (with positive probability), and the (positive) probability with which he submits a serious bid in each of those contingencies. We shall refer to this as the bidder's *bid initiation* strategy.

Lemma 0. In any pBe:

- (i) A bidder will bid up to his valuation in the current round if a serious bid has previously been submitted during that round. The set of pBe outcomes (i.e., distributions over reserve price path, time of sale, winning bidder and selling price) is unaffected if we assume that bid initiation strategies depend on (at most) the current and previous reserve prices.
- (ii) *[Successive skimming]* If a bidder with valuation v finds it optimal to initiate serious bidding in round t, with current reserve price R_t and following history $h_{t-1} = (R_1, R_2, ..., R_{t-1})$, then a bidder with valuation $v' > v$ will find it strictly optimal to do so.

Given Lemma 0, we shall henceforth summarise the seller's posterior beliefs about the distribution over bidder values by the point v at which those beliefs truncate F. A proof of Lemma 0 may be found in the Appendix.

It remains to characterise the bid-initiation strategies and the equilibrium path of reserve prices. The argument, which closely follows that in MV97, is based on the following sequence of artificially truncated games:

$\mathcal{G}(j, v)$**:** the repeated auction game in which the seller begins with posterior v and is required to hold a \$1 reserve auction if the object remains unsold after j rounds. (The seller's payoff function in $\mathcal{G}(j, v)$ will be assumed not to condition on bidder values being bounded by v.)

Analysis of these truncated games proves useful as the seller's optimal continuation strategy, given any posterior, will always set a \$1 reserve in finite time – see Lemma 2.

As in MV97, let us initialise

$$\gamma_0(v) \equiv \gamma_{-1}(v) \equiv \{1\}$$

$$r_0(x) \equiv 1$$

and let Y_1 be the random variable equal to the highest valuation amongst $n-1$ bidders, with $F_{Y_1} \equiv F^{n-1}$ the distribution of Y_1 and $f_{Y_1} \equiv F'_{Y_1}$ the associated density.

Note that

$$\Pi_0(v) = \int_1^v \int_1^{X_1} nY_1 f(X_1)\, dF_{Y_1} dX_1 \tag{1}$$

is the seller's (equilibrium) payoff in $\mathcal{G}(0,v)$.

Now consider $\mathcal{G}(1,v)$. Define $r_1(x)$ to be a reserve price that makes a bidder of valuation x indifferent between initiating bidding in the first round of $\mathcal{G}(1,v)$ or waiting for the \$1 reserve auction next period. Since x is the marginal bid initiator in the current round, and since the marginal bidder earns a non-zero payoff only if x is the highest value amongst the bidders, the marginal bidder's decision is made on the assumption that the game will proceed to the next round with certainty if he decides to wait. However, unlike in MV97, if the marginal bidder were to initiate bidding in the current round, it is possible that he wins the auction but pays above the reserve price – lower value bidders immediately join the bidding once a serious bid is received. Thus:

$$[x - r_1(x)]\, F_{Y_1}(r_1(x)) + \int_{r_1(x)}^x (x - Y_1)\, dF_{Y_1} = \delta \int_1^x (x - Y_1)\, dF_{Y_1}$$

Equivalently:

$$r_1(x)\, F_{Y_1}(r_1(x)) + \int_{r_1(x)}^x Y_1 dF_{Y_1} = (1-\delta)\, x F_{Y_1}(x) + \delta \int_1^x Y_1 dF_{Y_1} \tag{2}$$

Lemma A1. The solution $r_1(x)$ to (2) is unique, continuous and strictly increasing in x. Moreover, $r_1(x) < x$ for $x > 1$.

Proof. Since

$$\int_1^x Y_1 dF_{Y_1} \leq (1-\delta)\, x F_{Y_1}(x) + \delta \int_1^x Y_1 dF_{Y_1} \leq x F_{Y_1}(x)$$

and the function

$$z F_{Y_1}(z) + \int_z^x Y_1 dF_{Y_1} \tag{3}$$

is continuous in $z \in [1, x]$, there is at least one solution to (2). By direct calculation, the function (3) is strictly increasing in $z \in [1, x]$, so there is at most one solution. Continuity of $r_1(x)$ is obvious.

To see that $r_1(x)$ is strictly increasing, note that

$$\begin{aligned}\frac{\partial}{\partial x}\left[\int_z^x Y_1 dF_{Y_1} \; - \; (1-\delta)\, x F_{Y_1}(x) \; - \; \delta \int_1^x Y_1 dF_{Y_1}\right] &= -(1-\delta)\, F_{Y_1}(x) \\ &< 0.\end{aligned}$$

Since (3) is strictly increasing in $z \in [1, x]$, it follows that $r_1(x)$ must increase to restore equality (2) when x increases.

That $r_1(x) < x$ for $x > 1$ is now obvious. □

By virtue of Lemma A1, we can think of the seller in $\mathcal{G}(1, v)$ choosing the cutoff value x rather than the reserve price.

If we let

$$p_j(x) \;=\; r_j(x)\, F_{Y_1}(r_j(x)) \;+\; \int_{r_j(x)}^x Y_1 dF_{Y_1} \qquad j = 0, 1 \tag{4}$$

we may express (2) as follows:

$$p_1(x) \;=\; (1-\delta)\, x F_{Y_1}(x) \;+\; \delta p_0(x) \tag{5}$$

Note that $p_j(x)$ is the (unconditional) expected price that a bidder with value x anticipates paying if they win an auction with reserve price $r_j(x)$. Since (3) is a strictly increasing function of $z \in [1, x]$, there is a one-to-one mapping from $r_j(x)$ to $p_j(x)$. This allows us to re-write (1) as follows:

$$\Pi_0(v) \;=\; \int_1^v n f(X_1)\, p_0(X_1)\, dX_1.$$

Next, we define

$$g_1(v, x) \;=$$

$$r_1(x)\, n F_{Y_1}(r_1(x))\, [F(v) - F(x)] \;+\; \int_x^v \int_{r_1(x)}^{X_1} n Y_1 f(X_1)\, dF_{Y_1} dX_1 \;+\; \delta \Pi_0(x)$$

This is the payoff to the seller in $\mathcal{G}(1, v)$ when she chooses x to be the marginal bid initiator in the current round. Using (4), it may be expressed as follows:

$$g_1(v, x) \;=\; n \int_x^v \left[p_1(x) + \int_x^{X_1} Y_1 dF_{Y_1}\right] f(X_1)\, dX_1 \;+\; \delta \Pi_0(x) \tag{6}$$

Lemma 1. There exists an $\varepsilon > 0$ such that $r_1(v) = 1$ for all $\delta \in (0,1)$, all n and all $v < 1 + \varepsilon$.

Proof. We shall show that there is some $\varepsilon > 0$ such that

$$\frac{\partial g_1(v,x)}{\partial x} < 0$$

whenever $1 < x \leq v < 1 + \varepsilon$. Note that, using (6),

$$\frac{\partial g_1(v,x)}{\partial x}$$

$$\begin{aligned}
&= -np_1(x)\,f(x) \;+\; n\int_x^v \left\{\frac{\partial p_1(x)}{\partial x} - xf_{Y_1}(x)\right\} f(X_1)\;dX_1 \;+\; \delta\Pi_0'(x) \\
&= -nf(x)(1-\delta)\,xF_{Y_1}(x) \;+\; n[F(v) - F(x)]\left\{\frac{\partial p_1(x)}{\partial x} - xf_{Y_1}(x)\right\} \\
&= -nxf(x)(1-\delta)\,F_{Y_1}(x) \;+\; n[F(v) - F(x)](1-\delta)\,F_{Y_1}(x)
\end{aligned}$$

where the second equality uses (5) and

$$\Pi_0'(x) \;=\; nf(x)\int_1^x Y_1 dF_{Y_1}$$

and the last uses (5). Thus:

$$\frac{\partial g_1(v,x)}{\partial x} \;=\; (1-\delta)\,nF_{Y_1}(x)\,[F(v) - F(x) - xf(x)]$$

This matches the expression on p.266 of MV97, despite the differences between our expression for $g_1(v,x)$ and their analogous expression for $g_1(v,x,1)$ (see p.256). The Lemma therefore follows by the MV97 argument. $\square$

We now prove that the original game ends in finitely many rounds along any pBe path.

Lemma 2. There exists some $T < \infty$ such that the seller's pBe continuation strategy for $\mathcal{G}$, given any belief v, will set a \$1 reserve in no more than T further rounds.

Proof. If $v = v_H$ then it is obvious that v must fall below v_H next period when the seller follows an optimal strategy. Let $\overline{v} < v^H$ be (an upper bound on) this subsequent posterior. Suppose, therefore, that the seller's current posterior is $v \leq \overline{v}$, and that by following an optimal strategy this posterior will fall by z if the good remains unsold after k subsequent periods. Let $\Pi(v)$ denote the seller's expected discounted revenue along the pBe continuation path. Then

$$\Pi(v) \leq n[F(v) - F(v-z)]\, v^H + \delta^k v^H$$

since $n[F(v) - F(v-z)]$ is an upper bound on the probability that the maximum bidder value lies in $[v-z, v]$ and v^H is an upper bound on the selling price. Since f is bounded, there exists $\overline{f} < \infty$ such that $f \leq \overline{f}$ on $[1, \overline{v}]$. Thus:

$$\Pi(v) \leq \left[nz\overline{f} + \delta^k\right] v^H$$

Note that this upper bound is independent of v. Furthermore, the bound goes to zero if $z \longrightarrow 0$ and $k \longrightarrow \infty$. The seller can guarantee a sale immediately by posting a \$1 reserve. From Lemma 1 we deduce the existence of some $\varepsilon > 0$ such that

$$\Pi(v) \geq F(v) \geq F(1+\varepsilon) > 0.$$

It follows that there is some $z > 0$ and some $k < \infty$ such that the seller's posterior falls by at least z after k periods, independently of the value of $v \leq \overline{v}$. The result now follows directly. $\square$

As in MV97 we define:

$$\Pi_1(v) = \max_{x \leq v} g_1(v, x)$$

$$\gamma_1(v) = \arg\max_{x \leq v} g_1(v, x)$$

Lemma B1. The function $\Pi_1(v)$ is continuous and non-decreasing.

Proof. Continuity follows by the Theorem of the Maximum, given continuity of $p_1(x)$ (Lemma A1). That $\Pi_1(v)$ is non-decreasing is obvious. $\square$

Lemma C1. The correspondence γ_1 is non-decreasing[8] and uhc. Moreover[9]

$$\gamma_1^*(v) = \max \gamma_1(v) < v$$

[8] That is, if $v' > v$, $x \in \gamma_1(v)$ and $x' \in \gamma_1(v')$, then $x' \geq x$.

[9] Note that the following maximum is well-defined, since uhc correspondences are compact-valued: Berge (1997, p.110).

for any $v > 1$.

Proof. That γ_1 is uhc follows by the Theorem of the Maximum, given continuity of $r_1(x)$ (Lemma A1).

To see that it is non-decreasing, suppose $v' > v$, $x \in \gamma_1(v)$, $x' \in \gamma_1(v')$ and $x' < x$. Then

$$g_1(v', x') \;\geq\; g_1(v', x)$$

and

$$g_1(v, x) \;\geq\; g_1(v, x')$$

from which we deduce

$$g_1(v', x') - g_1(v, x') \;\geq\; g_1(v', x) - g(v, x)$$

which is equivalent to

$$\begin{aligned} & n \int_v^{v'} \left[p_1(x') + \int_{x'}^{X_1} Y_1 dF_{Y_1} \right] f(X_1)\, dX_1 \\ \geq \;& n \int_v^{v'} \left[p_1(x) + \int_{x}^{X_1} Y_1 dF_{Y_1} \right] f(X_1)\, dX_1 \end{aligned} \tag{7}$$

But

$$p_1(z) + \int_z^{X_1} Y_1 dF_{Y_1} \;=\; r_1(z) F_{Y_1}(r_1(z)) \;+\; \int_{r_1(z)}^{X_1} Y_1 dF_{Y_1}$$

is strictly increasing in z, since $r_1(z)$ is strictly increasing in z (Lemma A1), as is (3). It follows that (7) is impossible.

Finally, from the proof of Theorem 1, we see that

$$\left. \frac{\partial g_1(v, x)}{\partial x} \right|_{x=v} \;<\; 0$$

when $v > 1$, so $\gamma_1^*(v) = v$ is excluded. □

While $\gamma_1(v)$ may not be a singleton, it is obvious that the seller in $\mathcal{G}(2, v)$ prefers that first-round bidders anticipate the highest credible reserve price in the second round to encourage them to bid earlier. Hence, if play in $\mathcal{G}(2, v)$ reaches the second round, we expect the equilibrium cutoff to be chosen according to $\gamma_1^*(v)$.

Next, consider $\mathcal{G}(2,v)$. Given $w \leq x \leq v$, let $r_j(x,w)$ be a reserve price that makes a bidder of valuation x indifferent between initiating bidding in the first round of $\mathcal{G}(2,v)$ or the second.[10] The variable w is the second-round marginal bid initiator anticipated by the first-round marginal bidder. Arguing as for $\mathcal{G}(1,v)$ we have:

$$r_2(x,w) F_{Y_1}(r_2(x,w)) + \int_{r_2(x,w)}^{x} Y_1 dF_{Y_1} =$$

$$(1-\delta) x F_{Y_1}(x) + \delta \left\{ r_1(w) F_{Y_1}(r_1(w)) + \int_{r_1(w)}^{x} Y_1 dF_{Y_1} \right\} \tag{8}$$

This is the analogue of the expression on p.255 of MV97 with $j = 2$. Unlike their expression, with an English auction format the variable w only affects the marginal bidder's decision through its impact on the reserve price path: if r_1 did not depend on w then neither would r_2 nor the identity of the first-round marginal bidder. If we let

$$p_2(x,w) = r_2(x,w) F_{Y_1}(r_2(x,w)) + \int_{r_2(x,w)}^{x} Y_1 dF_{Y_1}$$

we may express (8) as follows:

$$p_2(x,w) = (1-\delta) x F_{Y_1}(x) + \delta p_1(w) \tag{9}$$

Lemma A2. The solution $r_2(x,w)$ to (8) is unique. It is also continuous and strictly increasing in each argument. Finally, $r_2(x,w) < x$ for $x > w \geq 1$.

Proof. All parts of Lemma A2 apart from $r_2(x,w)$ being continuous and strictly increasing in w may be proved along the lines of Lemma A1. We omit the details. The remaining parts of the Lemma follow easily from (8),

[10] Lemma 0 shows that, in each round, the seller's optimal strategy will carve a "top slice" from $[1,v]$ – possibly the whole interval (i.e., by setting a \$1 reserve). It is also intuitive that the seller will never choose this slice to be empty, as delay is costly. Hence, in each period, the seller's equilibrium strategy either sets a \$1 reserve, or there is a marginal type $x < v$ who is indifferent between initiating bidding now or waiting. Since x will be the seller's posterior next period, this bidder will certainly bid in the next round. Hence, it suffices to consider the bidder who is indifferent between initiating bidding now or next period, since the marginal bidder today will never wait more than one round before bidding.

using the facts that $r_1(w)$ is strictly increasing in w (Lemma A1) and (3) is strictly increasing in z. $\square$

Next, we define:

$$g_2(v,x,w) = n\int_x^v \left[p_2(x,w) + \int_x^{X_1} Y_1 dF_{Y_1}\right] f(X_1)\, dX_1 + \delta\Pi_1(x) \quad (10)$$

and hence, using the fact that g_2 is continuous in (x,w),

$$\Pi_2(v) = \max_{x\le v,\ w\in\gamma_1(x)} g_2(v,x,w)$$

$$\gamma_2(v) = \arg\max_{x\le v}\left[\max_{w\in\gamma_1(x)} g_2(v,x,w)\right]$$

Lemma B2. The function $\Pi_2(v)$ is continuous and non-decreasing.

Proof. Continuity follows by the Theorem of the Maximum, given the properties of $p_1(x,w)$ established in Lemma A2 and continuity of $\Pi_1(v)$ (Lemma B1). That $\Pi_2(v)$ is non-decreasing is obvious. $\square$

Lemma C2. The correspondence γ_2 is non-decreasing and uhc. Moreover

$$\gamma_2^*(v) = \max\gamma_2(v) < v$$

for any $v > 1$.

Proof. To see that $\gamma^*(v) < v$ when $v > 1$, note that $\Pi_2(v) \ge \Pi_1(v)$. Since

$$g_2(v,v,w) = \delta\Pi_1(v)$$

the result follows.

To prove the rest of Lemma C2, define

$$h(v,x) = \max_{w\in\gamma_1(x)} g_2(v,x,w).$$

Using Lemmas B1 and C1 and the Theorem of the Maximum, we deduce that h is usc. Hence γ_2 is uhc by another application of the Theorem of the

Maximum. That γ_2 is non-decreasing follows by the argument in the proof of Lemma C1, with h in place of g_1. $\square$

Once again, if play in $\mathcal{G}(3,v)$ reaches the second round, we expect the equilibrium cutoff to be chosen according to $\gamma_2^*(v)$ to put maximal pressure on bidders in the first round.

Finally, consider $\mathcal{G}(j,v)$ for $j>2$. Given $w \leq x \leq v$, we define $r_j(x,w)$ to be a reserve price that makes a bidder of valuation x indifferent between initiating bidding in the first round of $\mathcal{G}(j,v)$ or the second. Then:

$$r_j(x,w)\, F_{Y_1}(r_j(x,w)) + \int_{r_j(x,w)}^{x} Y_1 dF_{Y_1} =$$

$$(1-\delta)\, x F_{Y_1}(x) + \delta \left\{ r_{j-1}\left(w, \gamma_{j-2}^*\right) F_{Y_1}\left(r_{j-1}\left(w,\gamma_{j-2}^*\right)\right) + \int_{r_{j-1}\left(w,\gamma_{j-2}^*\right)}^{x} Y_1 dF_{Y_1} \right\} \tag{11}$$

where γ_{j-2}^* is the equilibrium value of the marginal bid-initiator two periods hence. As usual, by defining

$$p_j(x,w) = r_j(x,w)\, F_{Y_1}(r_j(x,w)) + \int_{r_j(x,w)}^{x} Y_1 dF_{Y_1}$$

we may express (11) as follows:

$$p_j(x,w) = (1-\delta)\, x F_{Y_1}(x) + \delta p_{j-1}\left(w, \gamma_{j-2}^*\right)$$

Letting

$$g_j(v,x,w) = n \int_x^v \left[p_j(x,w) + \int_x^{X_1} Y_1 dF_{Y_1} \right] f(X_1)\, dX_1 + \delta \Pi_{j-1}(x)$$

$$\Pi_j(v) = \max_{x \leq v,\; w \in \gamma_{j-1}(x)} g_j(v,x,w)$$

$$\gamma_j(v) = \arg\max_{x \leq v} \left[\max_{w \in \gamma_{j-1}(x)} g_j(v,x,w) \right]$$

analogous arguments to those used in proving Lemmas A2–C2, plus the principle of induction, give, for each $j \geq 3$:

Lemma Aj. The solution $r_j(x,w)$ to (11) is unique. It is also continuous and strictly increasing in each argument. Finally, $r_j(x,w) < x$ for $x > w \geq 1$.

Lemma Bj. The function $\Pi_j(v)$ is continuous and non-decreasing.

Lemma Cj. The correspondence γ_j is non-decreasing and uhc. Moreover

$$\gamma_j^*(v) \;=\; \max \gamma_j(v) \;<\; v$$

for any $v > 1$.

Let us now construct the sequence $\{z_i\}_{i=0}^{\infty}$ as follows: $z_0 = 1$,

$$z_1 \;=\; \sup \{v \mid \gamma_1^*(v) = 1\}$$

and

$$z_i \;=\; \min \{\sup \{v \mid \gamma_i^*(v) < z_{i-1}\}\,,\; v_H\}$$

for $i > 1$. Combining Lemmas 1, 2 and Cj, there is some $T < \infty$ such that

$$1 = z_0 < z_1 < \cdots < z_T = v^H.$$

If we define $\Gamma_i = [z_{i-1}, z_i)$ for $i = 1, 2, ..., T-1$ and $\Gamma_T = [z_{T-1}, z_T]$, then $\gamma_j^*(v) < z_{j-1}$ when $v \in \Gamma_j$ and $j \geq 2$.

Lemma 3. For each $i \in \{1, 2, ..., T\}$, if $v \in \Gamma_i$ then $\gamma_j(v) = \gamma_i(v)$ for all j, and hence $\Pi_j(v) = \Pi_i(v)$ for all j also.

Proof. Since $\gamma_j^*(v) \leq v$ (Lemma Cj) a simple inductive argument implies that $\gamma_j^*(v) = 1$ for all $v \in \Gamma_1$ and all j. This proves the result for $i = 1$.

Consider $v \in \Gamma_2$. The seller in $\mathcal{G}(2,v)$ will choose a \$1 reserve after one further round, so her optimal strategy will be the same in $\mathcal{G}(1,v)$, implying $\gamma_1(v) = \gamma_2(v)$. Conversely, for $j > 2$, the seller in $\mathcal{G}(j,v)$ will set a \$1 reserve after no more than $j-1$ rounds, since $\gamma_j^*(x) \leq x$ (Lemma Cj). Thus, an inductive argument gives $\gamma_j(v) = \gamma_2(v)$.

Hence, by another layer of induction, we deduce the result. $\square$

We may therefore define γ, γ^* and Π on $\left[1, v^H\right]$ in the obvious fashion and

$$r(x,w) \;=\; \begin{cases} r_1(x) & \text{if } x \in \Gamma_1 \\ r_j(x,w) & \text{if } x \in \Gamma_j \text{ for } j > 1 \end{cases}$$

where $\gamma_{j-2}^* = \gamma^*(w)$ is used in the construction of $r_j(x, w)$. The argument on pp.266–269 of MV97 is now readily adapted to the present set-up to yield:[11]

Theorem 1. In any pBe of $\mathcal{G}$, the equilibrium strategies satisfy:

(a) $R_1 \in \{ r(v_2, \gamma^*(v_2)) \mid v_2 \in \gamma(v^H) \}$.

(b) Along the equilibrium path, for any $t > 1$

$$R_t = r(\gamma^*(v_t), \ \gamma^*(\gamma^*(v_t)));$$

(c) For any $t \geq 1$ and any history h_t, type x initiates bidding if $R_t < r(x, \gamma^*(x))$ and not if $R_t > r(x, \gamma^*(x))$.

Theorem 1 characterises the equilibrium reserve price path. As usual, there may be multiple first-period reserve prices in equilibrium, and the seller may even randomise over them. But once the first-period reserve has been chosen, all subsequent reserve prices along the equilibrium path are uniquely determined – see (b).

3 Revenue Equivalence and the Coase Conjecture

The objective of this section is to compare the expected discounted revenue generated by the repeated English auction to that generated by a repeated (first- or second-price) sealed-bid auction. The issue is not straightforward, as the optimal reserve price path for the repeated English auction does not match that of the repeated sealed-bid formats. As explained in the Introduction, bidders in a sealed-bid auction have incentives to initiate bidding earlier, *ceteris paribus*, because of the different participation effects. This raises the reserve price (relative to the English auction) necessary to make a type x bidder indifferent between bidding now or waiting. However, as we shall see, the revenue-optimal path of *bidder cutoffs* is the same across auction formats, as is the equilibrium expected discounted revenue.[12]

[11] We stress that the structure of the preceeding also follows MV97, though details differ. We have also attempted to elaborate somewhat on the arguments in MV97, and have occasionally reverted to those in Fundeberg, Levine and Tirole (1985), to add clarity.

[12] This fact is reminiscent of Proposition 2 in Riley and Zeckhauser (1983). Indeed, we conjecture that an analogue of their result holds with exponential discounting in place of the fixed sampling cost, c.

Define $\mathcal{G}^{MV}$ to be the repeated second-price, sealed-bid auction game from MV97. Likewise, define $\mathcal{G}^{MV}(j,v)$ to be the MV97 analogue of $\mathcal{G}(j,v)$ and $r_1^{MV}(x)$ the analogue of $r_1(x)$. Note that

$$\Pi_0^{MV}(v) = \Pi_0(v) = \int_1^v \int_1^{X_1} nY_1 f(X_1)\, dF_{Y_1} dX_1$$

is the seller's (equilibrium) payoff in $\mathcal{G}^{MV}(0,v)$. Moreover, the analogues of (2) and $g_1(v,x)$ are, respectively:

$$r_1^{MV}(x) F_{Y_1}(x) = (1-\delta) x F_{Y_1}(x) + \delta \int_1^x Y_1 dF_{Y_1} \tag{12}$$

and

$$g_1^{MV}(v,x) =$$
$$r_1^{MV}(x) nF_{Y_1}(x)[F(v) - F(x)] + \int_x^v \int_x^{X_1} nY_1 f(X_1)\, dF_{Y_1} dX_1 + \delta\Pi_0(x)$$

For the case $j = 1$, the truncated game $\mathcal{G}(j,v)$ has a higher equilibrium reserve price in the first round than $\mathcal{G}^{MV}(j,v)$, but the same bidder types initiate first-round bidding:

Lemma 4. If $x > 1$, then $r_1^{MV}(x) > r_1(x)$.

Proof. Obvious. □

Lemma 5. If $v \geq x \geq 1$, then $g_1^{MV}(v,x) = g_1(v,x)$.

Proof. From the proof of Lemma 1, we know that

$$\frac{\partial g_1^{MV}(v,x)}{\partial x} \equiv \frac{\partial g_1(v,x)}{\partial x}.$$

Since $r_1(1) = r_1^{MV}(1) = 1$, we also have

$$g_1^{MV}(v,1) = g_1(v,1)$$

and the result follows. □

Since we may write

$$g_1^{MV}(v,x) = n\int_x^v \left[p_1^{MV}(x) + \int_x^{X_1} Y_1 dF_{Y_1}\right] f(X_1)\, dX_1 \ + \ \delta\Pi_0(x)$$

where

$$p_1^{MV}(x) = r_1^{MV}(x) F_{Y_1}(x)$$

it follows from Lemma 5 that

$$n[F(v) - F(x)]\, p_1(x) \equiv n[F(v) - F(x)]\, p_1^{MV}(x)$$

and hence

$$p_1(x) + \int_x^{X_1} Y_1 dF_{Y_1} = p_1^{MV}(x) + \int_x^{X_1} Y_1 dF_{Y_1}$$

for any $X_1 \in [x, v]$. In other words, if the seller chooses cutoff x in the first round of $\mathcal{G}(1, v)$ and type X_1 wins the (first-round) auction, then the expected payment to the seller is the same across formats. This suggests a revenue-equivalence result. The English auction format provides less incentive for bidders to bid early, so the seller must choose a *lower* reserve price to induce the same set of types to initiate first-period bidding. However, this is exactly offset by the fact that, in the English auction, types below x participate once bidding starts.

More generally, we have:

Theorem 2. For any pBe of $\mathcal{G}$, there is a revenue-equivalent pBe of $\mathcal{G}^{MV}$ and conversely.

Proof. It suffices to show that (in obvious notation):

$$\gamma_j^{MV}(v) \equiv \gamma_j(v)$$

$$z_j^{MV} = z_j$$

and

$$\Pi_j^{MV}(v) \equiv \Pi_j(v).$$

for each $j \geq 1$. We shall argue by induction.

The stated equalities hold for $j = 1$ by Lemma 5. We now show that

$$\frac{\partial g_2^{MV}(v,x,w)}{\partial x} \equiv \frac{\partial g_2(v,x,w)}{\partial x}$$

and hence

$$g_2^{MV}(v,x,w) \equiv g_2(v,x,w).$$

The quantity $r = r_2(x,w)$ satisfies

$$F_{Y_1}(x)\{x - \mathbb{E}[\max\{r, Y_1\} \mid Y_1 \le x]\} = \delta\left\{xF_{Y_1}(x) - \left[p_1(w) + \int_w^x Y_1 dF_{Y_1}\right]\right\}$$

which implies

$$n(1-\delta)xf(x)F_{Y_1}(x) =$$

$$nf(x)F_{Y_1}(x)\mathbb{E}[\max\{r, Y_1\} \mid Y_1 \le x] - \delta nf(x)\left[p_1(w) + \int_w^x Y_1 dF_{Y_1}\right]$$

The right-hand side of this expression is the *reduction* in $g_2(v,x,w)$ due to the *direct* effect of dx: the probability that $X_1 = x$, multiplied by the change in the discounted expected payment from selling now rather than in the next round. Likewise:

$$n(1-\delta)xf(x)F_{Y_1}(x) = nf(x)F_{Y_1}(x)r - \delta nf(x)\left[p_1^{MV}(w) + \int_w^x Y_1 dF_{Y_1}\right]$$

when $r = r_2^{MV}(x,w)$. Hence the direct effects are the same across the two auction formats.

It is easy to check that the *indirect* effect of dx – the probability of sale at a price equal to $r(x)$ times $r'(x)$ – is the same across both auctions and equal to $(1-\delta)n[F(v) - F(x)]F_{Y_1}(x)$.

The rest of the induction proceeds similarly. $\square$

It is clear from the proof of Theorem 2 that, for two revenue-equivalent pBe's, the equilibrium reserve price path in the English auction model lies *below* that of the sealed-bid model (and strictly so unless $r = r^{MV} = 1$).

To help develop intuition for Theorem 2, let us sketch an alternative proof, inspired by Stokey (1979). Consider the game $\mathcal{G}(j,v)$. Suppose the seller chooses a sequence $x \equiv \{x_t\}_{t=1}^{j+1}$ of type cut-offs satisfying

$$1 = x_{j+1} \le x_j \le x_{j-1} \le \cdots \le x_1 \le v \tag{13}$$

Types in (x_t, x_{t-1}) initiate bidding in period t. Type x_t is indifferent between bid initiation in period t or $t+1$. To implement this price discrimination strategy, the seller may use the non-increasing reserve price sequence $\{R_t(x)\}_{t=1}^{j+1}$

with $R_{j+1}(x) = 1$ and $R_t(x) = r(x_t, x_{t+1})$ for $t \le j$. Define

$$p_t(v') = \frac{1}{F_{Y_1}(v')} \left\{ R_t(x) F_{Y_1}(R_t(x)) + \int_{R_t(x)}^{v'} Y_1 dF_{Y_1} \right\}$$

to be the expected sale price conditional on type $v' \le v$ winning the auction in round t. Then the seller's expected discounted payoff from (13) is:

$$\delta^{j+1} + \sum_{k=0}^{j-1} \int_{x_{j+1-k}}^{x_{j-k}} \left\{ [1 - F^n(X_1)] \delta^{j+1-k} \left[\frac{d}{dX_1} p_{j+1-k}(X_1) \right] \right\} dX_1 + \sum_{k=0}^{j-1} [1 - F^n(x_{j-k})] \Delta_k(x) \quad (14)$$

where

$$\Delta_k(x) = \delta^{j-k} p_{j-k}(x_{j-k}) - \delta^{j+1-k} p_{j+1-k}(x_{j-k})$$

is the difference in the discounted (conditional) expected price between selling to type x_{j-k} in period $j-k$ or in period $j+1-k$.

Similarly, the seller in $\mathcal{G}^{MV}(j, v)$ may implement the sequence $x \equiv \{x_t\}_{t=1}^{j+1}$ satisfying (13) using the non-increasing reserve price sequence $\left\{R_t^{MV}(x)\right\}_{t=1}^{j+1}$ with $R_{j+1}^{MV}(x) = 1$ and $R_t^{MV}(x) = r^{MV}(x_t, x_{t+1})$ for $t \le j$. In this case, the expected sale price conditional on type $v' \le v$ winning the auction in round t is

$$p_t^{MV}(v') = \frac{1}{F_{Y_1}(v')} \left\{ R_t^{MV}(x) F_{Y_1}(x_t) + \int_{x_t}^{v'} Y_1 dF_{Y_1} \right\}$$

This implies seller payoff

$$\delta^{j+1} + \sum_{k=0}^{j-1} \int_{x_{j+1-k}}^{x_{j-k}} \left\{ [1 - F^n(X_1)] \delta^{j+1-k} \left[\frac{d}{dX_1} p_{j+1-k}^{MV}(X_1) \right] \right\} dX_1 + \sum_{k=0}^{j-1} [1 - F^n(x_{j-k})] \Delta_k^{MV}(x) \quad (15)$$

where

$$\Delta_k^{MV}(x) = \delta^{j-k} p_{j-k}^{MV}(x_{j-k}) - \delta^{j+1-k} p_{j+1-k}^{MV}(x_{j-k})$$

We shall show that (14) and (15) are the same. First, for any k:

$$\Delta_k(x) = \Delta_k^{MV}(x) = (1-\delta)\,\delta^{j-k}x_{j-k} \tag{16}$$

by the definition of x_{j-k}. In other words, for x_{j-k} to be indifferent between bidding in periods $j-k$ or $j+1-k$, the additional waiting cost must exactly balance the expected reduction in discounted price paid. Next

$$\begin{aligned} & \frac{d}{dX_1}\left\{R_t(x)\,F_{Y_1}(R_t(x)) + \int_{R_t(x)}^{X_1} Y_1 dF_{Y_1}\right\} \\ = \;& \frac{d}{dX_1}\left\{R_t^{MV}(x)\,F_{Y_1}(x_t) + \int_{x_t}^{X_1} Y_1 dF_{Y_1}\right\} \\ = \;& X_1 F'_{Y_1}(X_1) \end{aligned}$$

so the marginal effect on the *unconditional* expected selling price from a marginal increase in X_1 is the same across auction formats. Given that (14) and (15) start from a common minimum selling price (\$1), and given the equality in (16), it follows that the marginal effects on the *conditional* prices are also the same:

$$\frac{d}{dX_1}p_{j+1-k}^{MV}(X_1) = \frac{d}{dX_1}p_{j+1-k}^{MV}(X_1)$$

for all $X_1 \in (x_{j+1-k}, x_{j-k})$ and all k. This proves the equality of (14) and (15).

Therefore, the seller's payoff from implementing $x \equiv \{x_t\}_{t=1}^{j+1}$ is the same across auction formats. Therefore, by characterising the seller's problem as one of choosing type cut-offs rather than reserve prices, we see that the set of pBe cut-offs will coincide across auction formats, implying Revenue Equivalence.

The foregoing argument clearly suggests that there is a more general Revenue Equivalence Theorem at work. That is, by adapting the arguments in Myerson (1981) to our dynamic setting, it is possible to show that any two selling mechanisms that allocate the object in the same way – the probability that buyer i receives the object in period t when his type is v_i is the same for any i, t and v_i – and such that a buyer with $v_i = 1$ receives the same expected utility, will generate the same expected revenue for the seller.[13] In

[13] The interested reader may refer to the working paper version, which is available by request from the authors.

the present paper, we instead focus on revenue equivalence as an intermediate step towards proving the Coase conjecture for the case of English auctions, which can be stated as follows:

Theorem 3. In any pBe of $\mathcal{G}$, the seller's expected discounted revenue approaches that of a one-shot, no reserve auction as $\delta \to 1$.

This result follows directly from our Theorem 2 and Theorem 3 in MV97.

4 Appendix

Proof of Lemma 0:

Proof of (i). The first part has already been proved. From this it follows that *a given bidder expects to win the object if, and only if, he has the highest valuation*, no matter what bid-initiation strategy that he – or anyone else – uses.[14] This relies on the fact that bidders in an English auction can observe when serious bidding starts. A bidder's optimal bid-initiation strategy would therefore remain optimal were he to be (reliably) informed that his was the highest value.

The *optimality* of initiating bidding can only depend on history through the current and previous reserve prices. It cannot depend on the probabilities with which the bidder has mixed in the past, since these are not observable to the seller or to the other bidders. Likewise, it cannot depend on the elapsed time since the start of round t. It would do so only if the seller's or some rival bidder's continuation strategy were contingent on this elapsed time. But the seller's continuation strategy is relevant only if round t ends without a serious bid being received: plans formed by the seller at earlier points in time during the round are redundant, and may be assumed to coincide with her plan at the end of the round. Since the optimality of bid *initiation* at t is established within the contingency in which no other bidder bids in round t, the optimality of the decision is unaffected by anything learned about rival bidders through the elapse of time within round t.

Hence, *whether or not it is optimal* for a bidder with valuation v to initiate bidding in round t will depend only on the current reserve price R_t

[14]The "if" part of this observation ignores the zero measure event in which two or more bidders tie for highest valuation.

and history h_{t-1} of previous reserve prices. However, if a bidder is indifferent between bid initiation at t and waiting, his decision may be made contingent on elapsed time since the start of round t or on previous mixing by the bidder himself without violating optimality. Nevertheless, such a strategy can always be replaced with an "equivalent" one that depends only on the reserve price history. In particular, the seller cares only about the probability that a bidder with value v initiates serious bidding in round t given R_t and h_{t-1}. (She cannot observe bidders' mixing probabilities.) These bid-initiation probabilities can be computed from the bidders' strategies, under suitable technical restrictions on bidding strategies to ensure measurability. If bidders were to amend their pBe strategies to depend only on R_t and h_{t-1} and to conform to the seller's expectations, the pBe would not be disturbed. In particular, the seller's strategy remains optimal, and bidders would still be initiating serious bidding only when it is optimal to do so, this optimality being determined by R_t and h_{t-1} alone, given the seller's strategy. It is also clear that the pBe outcome is unaffected by the change.

Proof of (ii). Define $U(v, h_t)$ to be the *pBe continuation value* of a bidder with value v if the good remains unsold at the end of round t following reserve price history h_t.[15] Note that $U(v, h_t)$ is necessarily independent of the identity of the bidder in the case of $n > 1$. The difference

$$U(v', h_t) - U(v, h_t) \tag{17}$$

is *bounded above* by $\delta(v' - v)$, since the v type can imitate the v' type's strategy.

For $n = 1$, the result now follows easily, as shown by Fudenberg, Levine and Tirole (1985, Lemma 1). The difference between the payoffs (i.e., the v' return less the v return evaluated at the start of round t) from bidding in t is $\delta(v' - v)$, since the selling price is fixed at R_t and the sale occurs one unit of time hence. Since the difference (17) must be discounted by $\delta < 1$ when evaluated from the perspective of the start of round t, the v' type has a *strictly stronger relative incentive* to bid now rather than later.

[15] We define $h_t = h_{t-1}|R_t$ where $x|y$ denotes the concatenation of the vectors $x = (x_1, ..., x_m)$ and $y = (y_1, ..., y_k)$:

$$x|y = (x_1, ..., x_m, y_1, ..., y_k).$$

When $n > 1$ matters are more complex, as the expected sale price depends on the valuation of the winning bidder.

With *sealed bid* auctions, and allowing for the possibility that bidders may use *mixed* strategies, a bidder may not know for sure how many rivals he will face if he bids at t, nor whether the good will remain unsold at the start of round $t+1$ if he refrains from bidding at t. Thus, the difference in future returns (17) is now discounted by $\delta\rho_t(h_t)$, where $\rho_t(h_t)$ is the probability that all other bidders refrain from bidding at t. Next, consider the difference in returns from bidding at t. With probability $\rho_t(h_t)$, this difference is just $\delta(v' - v)$, since no-one else bids. Likewise if the highest rival bid is below v. In all other scenarios, the difference is some non-negative quantity, so the expected difference is *bounded below* by

$$\rho_t(h_t)\,\delta\,(v' - v)$$

and *strictly* so if $\rho_t(h_t) = 0$. Since

$$\rho_t(h_t)\,\delta \;\geq\; \delta^2\rho_t(h_t)$$

with equality iff $\rho_t(h_t) = 0$, the result follows by the same logic as in Fudenberg, Levine and Tirole. This is the gist of the argument in McAfee and Vincent (1997, Lemma 0).

Finally, for repeated *English* auctions, matters are more complex still. Now the bid-initiation decision may be made *conditional* on the event that the object remains unsold at the start of $t+1$ if the decision is to refrain from initiating bidding at t. Thus, difference (17) in continuation payoffs are once again discounted by δ. The difference in expected return from bidding now is[16]

$$\delta\left[G_{h_t}^{n-1}(v)\,(v' - v) \;+\; \int_v^{v'} (v' - x)\;dG_{h_t}^{n-1}(x)\right]$$

where G_{h_t} is the distribution of types conditional on no rival submitting a bid up to the end of round t given h_t. It is not possible to assert that this quantity exceeds $\delta^2(v' - v)$.

To prove the result, we must therefore "unpack" $U(v, h_t)$ to obtain a tighter upper bound on (17). We shall here sketch an argument that does so. The basic idea is straightforward. Fix a particular bidder – say bidder 1. Imagine this bidder facing a deterministic path of reserve prices R_s, $s =$

[16] The following expression assumes $v' > v \geq R_t$.

$t, t+1, t+2, ...$ Let Y_1 be the random variable denoting the highest valuation amongst bidder 1's rivals, and let $Z(s) = \max\{R_s, Y_1\}$. Let G_s denote the distribution of $Z(s)$ conditional on $v \geq Y_1$ and the object remaining unsold at the start of round s, given the equilibrium expectations of bidder 1 at the start of round t.

Let $s \geq t$ and consider the difference:

$$\int_1^v (v-z)\, dG_t(z) \;-\; \delta^{s-t} \int_1^v (v-z)\, dG_s(z) \tag{18}$$

$$= \; v\left[G_t(v) - \delta^{s-t} G_s(v)\right] \;-\; \left[\int_1^v z dG_t(z) \;-\; \delta^{s-t} \int_1^v z dG_s(z)\right]$$

The derivative of this expression with respect to v is

$$G_t(v) - \delta^{s-t} G_s(v) \;>\; 0$$

where the inequality follows since $\delta < 1$ and $G_t(v) = G_s(v) = 1$. If bidder 1 contemplates bid initiation in period $t' > t$, then his payoff will be a weighted average of terms

$$\delta^k \int_1^v (v-z)\, dG_{t+k}(z)$$

for $k = 0, 1, ..., t' - t$, since others may initiate bidding before t'. Thus, the relative merits of bidding at t rather than t' will therefore be represented by a weighted sum of terms of the form (18) for $s = 0, 1, ..., t'$, with the weights corresponding to probabilities that the bidding process starts in round s. It follows that the relative incentive to bid now is increasing in v.

If bidder 1 faces a random path of future reserve prices, we may simply apply the foregoing analysis to each possible path to reach the same conclusion. □

5 References

Ausubel, L.M., P. Crampton and R.J. Deneckere (2002): "Bargaining with Incomplete Information," in the *Handbook of Game Theory, Volume 3* (R.J. Aumann and S. Hart, Eds), pp.1897-1945. Elsevier Science.

Beckert, W. (2006): "Competitive Externalities in Dynamic Monopolies with Stochastic Demand" *Topics in Theoretical Economics* 69(1), Article 17.

Berge, C. (1997): *Topological Spaces.* Mineola, NY: Dover.

Coase, R. (1972): "Durability and Monopoly," *Journal of Law and Economics* 15, 143-149.

Fudenberg, D., D. Levine and J. Tirole (1985): "Infinite Horizon Models of Bargaining with Incomplete Information," in *Game Theoretic Models of Bargaining* (A. Roth, Ed.), pp.73-98. London/New York: CUP.

Grant, S., A. Kajii, F. Menezes and M.J. Ryan (2006): "Auctions with Options to Re-Auction," *International Journal of Economic Theory* 2(1), 17-39.

McAfee, R.P. and D. Vincent (1997): "Sequentially Optimal Auctions," *Games and Economic Behavior* 18, 246-276.

Myerson, R.B. (1981): "Optimal Auction Design," *Mathematics of Operations Research* 6(1), 58-73.

Riley, J. and R. Zeckhauser (1983): "Optimal Selling Strategies: When to Haggle and When to Hold Firm," *Quarterly Journal of Economics* 98(2), 267-289.

Rothschild, M. and J. E. Stiglitz (1970): "Increasing Risk I: A Definition," *Journal of Economic Theory* 2, 225–243

Stokey, N.L. (1979): "Intertemporal Price Discrimination," *Quarterly Journal of Economics* 93(3), 355-371.

Tirole, J. (1988): *Theory of Industrial Organization.* Cambridge, MA: MIT Press.